

SQUAREFREE NUMBERS IN ARITHMETIC PROGRESSIONS

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ABSTRACT. We give asymptotics for correlation sums linked with the distribution of squarefree numbers in arithmetic progressions over a fixed modulus. As a particular case we improve a result of Blomer [1] concerning the variance.

1. INTRODUCTION

For a positive real number X and positive integers a, q with $(a, q) = 1$, let $E(X, q, a)$ be defined by the formula

$$(1.1) \quad \sum_{\substack{n \leq X \\ n \equiv a \pmod{q}}} \mu^2(n) = \frac{6}{\pi^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{X}{q} + E(X, q, a),$$

where, as usual, μ is the Möbius function. In (1.1) the first term heuristically appears to be a good approximation of the number of squarefree integers $\leq X$ congruent to $a \pmod{q}$. In this paper, we are concerned with the so called error term $E(X, q, a)$. Trivially, one has

$$(1.2) \quad |E(X, q, a)| \leq \frac{X}{q} + 1,$$

while in [7], Hooley proved

$$(1.3) \quad E(X, q, a) = O_\epsilon \left(\left(\frac{X}{q} \right)^{1/2} + q^{1/2+\epsilon} \right),$$

where the O_ϵ -constant depends only on $\epsilon > 0$ arbitrary. This is the best result available for fixed a . Furthermore, formula (1.3) gives an asymptotic formula for the left-hand side of (1.1) for $q \leq X^{\frac{2}{3}-\epsilon}$ (We believe that such an asymptotic formula should hold for $q \leq X^{1-\epsilon}$ and it is a challenging problem to go beyond $X^{\frac{2}{3}-\epsilon}$ for a general q , in particular when q is prime). The situation is quite similar to the equivalent problem for the divisor function $d(n)$, which is, roughly, justified by the appearance of exponential sums on it.

While for the divisor function we consider the classical Kloosterman sums

$$K(a, b; q) := \sum_{\substack{1 \leq x \leq q-1 \\ (x, q)=1}} e \left(\frac{ax + b\bar{x}}{q} \right),$$

for the μ^2 function, we have to deal with

$$K_2(a, b; q) := \sum_{\substack{1 \leq x \leq q-1 \\ (x, q)=1}} e \left(\frac{ax + b\bar{x}^2}{q} \right),$$

where $e(x) = e^{2\pi i x}$ and \bar{x} is the multiplicative inverse of $x \pmod{q}$.

Blomer [1] considered a certain average over the residue classes $(\bmod q)$. More precisely, he considered the following second moment(variance) of the $E(X, q, a)$

$$(1.4) \quad \mathcal{M}_2(X, q) = \sum_{a \pmod{q}}^* |E(X, q, a)|^2,$$

where the $*$ symbol means we only sum over the classes that are relatively prime to q . In [1, Theorem 1.3], he showed that

$$(1.5) \quad \mathcal{M}_2(X, q) \ll X^\epsilon \left(X + \min \left(\frac{X^{5/3}}{q}, q^2 \right) \right)$$

holds for every $\epsilon > 0$, uniformly for $1 \leq q \leq X$.

Several years before, Croft [3] considered a variation of $\mathcal{M}_2(X, q)$ by summing not only over the classes relatively prime to q but over all the classes $(\bmod q)$. Let

$$\mathcal{M}_2'(X, q) = \sum_{a \pmod{q}} \left\{ \sum_{\substack{n \leq X \\ n \equiv a \pmod{q}}} \mu^2(n) - \frac{\mu^2(d)q_0}{\varphi(q_0)} \frac{6}{\pi^2} \prod_{p|q} (1 + p^{-1})^{-1} \frac{X}{q} \right\}^2,$$

where $d = (a, q)$ and $q_0 = q/d$. The last term between the curly brackets on the expression above can be seen as the expected value of the sum

$$\sum_{\substack{n \leq X \\ n \equiv a \pmod{q}}} \mu^2(n)$$

when a is not necessarily relatively prime to q . Observe that it reduces to the first term on the right-hand side of (1.1) whenever $(a, q) = 1$.

By doing an extra average over q , Croft [3, Theorem 2] proved the following formula

$$(1.6) \quad \sum_{q \leq Q} \mathcal{M}_2'(X, q) = BX^{1/2}Q^{3/2} + O \left(X^{2/5}Q^{8/5} \log^{13/5} X + X^{3/2} \log^{7/2} X \right), \quad (Q \leq X)$$

where B is an explicit constant. For $Q \geq X^{2/3+\epsilon}$, (1.6) above gives us an asymptotic formula for

$$\sum_{q \leq Q} \mathcal{M}_2'(X, q)$$

which looks like the classical Barban-Davenport-Halbertam for primes with a main term (see Montgomery [8]). This result was further improved by several authors. We mention the works of Warlimont [17], Brüdern *et al.* [2] and Vaughan [15]. It is worth mentioning that the later two results deal with the more general case of k -free numbers.

Ignoring for the moment the difference between $\mathcal{M}_2'(X, q)$ and $\mathcal{M}_2(X, q)$, formula (1.6) above can be interpreted as saying that, at least on average over $q \leq Q$, the value of $\mathcal{M}_2(X, q)$ is much smaller than the bound given by (1.5). In this paper we investigate if such phenomenon can be observed without the need of the extra average over q . This is indeed possible for q large enough relatively to X . Our main theorem goes in that direction

Theorem 1.1. *Let $\mathcal{M}_2(X, q)$ be defined as in (1.4) and $\epsilon > 0$ arbitrary. Then, uniformly for $q \leq X$, we have*

$$(1.7) \quad \mathcal{M}_2(X, q) = C \prod_{p|q} \left(1 + 2p^{-1} \right)^{-1} X^{1/2} q^{1/2} + O \left(d(q) X^{1/3} q^{2/3} + X^{23/15} q^{-13/15} (\log X)^{15} \right),$$

where

$$(1.8) \quad C = \frac{\zeta\left(\frac{3}{2}\right)}{\pi} \prod_p \left(\frac{p^3 - 3p + 2}{p^3} \right) = 0.167 \dots$$

and the O -constant is absolute.

As pointed out to the author by Professor Rudnick, the same constant was found by Hall [5] on the main term of the second moment for the problem of squarefree numbers in short intervals. Our technique is more general since we can also provide a glimpse on the correlation between $E(X, q, a)$ and $E(X, q, ma)$ for fixed m with $(m, q) = 1$. For $X > 0$, m, q integers such that $m \neq 0$, $(m, q) = 1$, $q \geq 1$ we define

$$(1.9) \quad \mathcal{M}_2[m](X, q) = \sum_{a \pmod{q}}^* E(X, q, a) E(X, q, ma).$$

The Theorem 1.1 above can be deduced as a consequence of the following

Theorem 1.2. *Let m be a nonzero integer of arbitrary sign. Also let $\mathcal{M}_2[m](X, q)$ be defined as in (1.9). Then, uniformly for $q \leq X$, $(m, q) = 1$ we have*

$$(1.10) \quad \mathcal{M}_2[m](X, q) = \frac{C}{2} \Gamma_{\text{an}}(m) \Gamma_{\text{ar}}(m) \prod_{p|q} \left(1 + 2p^{-1} \right)^{-1} X^{1/2} q^{1/2} \\ + O_m \left(d(q) X^{1/3} q^{2/3} + X^{23/15} q^{-13/15} (\log X)^{15} \right),$$

where C is as in (1.8), where the analytic factor $\Gamma_{\text{an}}(m)$ is defined by

$$(1.11) \quad \Gamma_{\text{an}}(m) := \begin{cases} \frac{\sqrt{m} + 1 - \sqrt{m-1}}{m}, & \text{if } m > 0, \\ \frac{\sqrt{1-m} - \sqrt{-m} - 1}{-m}, & \text{if } m < 0, \end{cases}$$

and the arithmetic factor $\Gamma_{\text{ar}}(m)$ is

$$(1.12) \quad \Gamma_{\text{ar}}(m) := \prod_{p^2|m} \left(-\frac{p^2-1}{p+2} \right) \prod_{\substack{p|m \\ p^2 \nmid m}} \left(1 + \frac{p+p^{1/2}+1}{p^{3/2}+p^{1/2}+1} \right)^{-1};$$

and the O_m -constant depends at most on m .

We state the result for general m but we content ourselves with a complete proof only for m squarefree.

Notice that making the choice $m = 1$, one has $\Gamma_{\text{an}}(1) = 2$, $\Gamma_{\text{ar}}(1) = 1$ and we retrieve Theorem 1.1. If one seeks for uniformity in m , the same techniques give an error term of the same order for $m \leq (\log X)^A$ for some constant A .

Notice that for $X^{8/13+\epsilon} < q < X^{1-\epsilon}$, formula (1.7) improves the upper bound (1.5). We have further the following direct consequence of Theorem 1.1

Corollary 1.3. *Let $\mathcal{M}_2(X, q)$ be defined as in (1.4) and $\epsilon > 0$ arbitrary. Then as $X \rightarrow \infty$, we have*

$$(1.13) \quad \mathcal{M}_2(X, q) \sim C \prod_{p|q} \left(1 + 2p^{-1} \right)^{-1} X^{1/2} q^{1/2}$$

uniformly for $X^{31/41+\epsilon} \leq q \leq X^{1-\epsilon}$ and where C is as in (1.8).

Remark 1.1. *The asymptotic formula (1.13) gives an average order of magnitude of $(X/q)^{\frac{1}{4}+\epsilon}$ for the terms $E(X, q, a)$. This remark goes in the direction of the conjecture due to Montgomery (see [3, top of the page 145]), which we write under the form*

$$E(X, q, a) = O_{\epsilon} \left((X/q)^{\frac{1}{4}+\epsilon} \right), \epsilon > 0 \text{ arbitrary}$$

uniformly for $(a, q) = 1$, $X^{\theta_1} < q < X^{\theta_2}$ where the values of the constants θ_1 and θ_2 satisfying $0 < \theta_1 < \theta_2 < 1$ have to be precised.

Note that the same phenomenon can be observed in the work of Croft, with an extra average over q . Such a conjecture may, of course, be interpreted in terms of the poles of the functions $\frac{L(\chi, s)}{L(\chi^2, 2s)}$, where χ is a Dirichlet character modulo q . The error term $O(X^{23/15+\epsilon} q^{-13/15})$ in (1.7) comes from the use of the square sieve (see [6]). The proof could be simplified by avoiding the use of this sieve, obtaining a worse error term. One can obtain $O(X^{5/3+\epsilon} q^{-1})$ in a rather elementary way. That would imply that equivalence (1.13) would only hold for $X^{7/9+\epsilon} \leq q \leq X^{1-\epsilon}$.

1.1. Discussion about $\Gamma_{\text{an}}(m)$ and $\Gamma_{\text{ar}}(m)$.

Consider the case where m is squarefree. Formula (1.10) shows that the random variables

$$\mathbf{X} : a \pmod{q} \mapsto \frac{E(X, q, a)}{(X/q)^{1/2}} \text{ and } \mathbf{X}_m : a \pmod{q} \mapsto \frac{E(X, q, ma)}{(X/q)^{1/2}}$$

are not asymptotically independent as $X \rightarrow \infty$, q satisfying $X^{31/41+\epsilon} \leq q \leq X^{1-\epsilon}$. This is an easy consequence of the fact that these random variables have asymptotic mean equal to zero (see lemma 8.1 below). The fact that \mathbf{X} and \mathbf{X}_m are dependent (when $m > 0$) can be guessed in a similar way as in [4, remark 1.8], by the trivial fact that for a squarefree n such that

$$n \equiv a \pmod{q}, 1 \leq n \leq X/m, (n, m) = 1,$$

then $n' = mn$ satisfies

$$n' \text{ squarefree}, n' \equiv ma \pmod{q}, 1 \leq n' \leq X.$$

Such an interpretation obviously fails when $m < 0$ or m is not squarefree, which may explain the signs of $\Gamma_{\text{an}}(m)$ and $\Gamma_{\text{ar}}(m)$. We also remark that the random variable

$$\frac{1}{\phi(q)} \sum_{m \pmod{q}}^* \mathbf{X} \mathbf{X}_m$$

has asymptotic mean zero, again by lemma 8.1.

Finally we would like to point out some differences between Theorem 1.2 and [4, Theorem 1.5] from which this study was inspired.

From [4, Corollary 1.7.], the correlation for the divisor function exists if and only if $m > 0$, with a correlation coefficient that is always positive. While in our setting, the correlation always exists and the sign depends on the value of m , corresponding to the sign of m , for m squarefree.

In the work of Fouvry *et al.*[4] instead of only considering a homothety $a \mapsto ma$, they consider general Möbius transformations

$$a \mapsto \gamma(a) := \frac{m_1 a + m_2}{m_3 a + m_4}.$$

The case of a general linear map $\gamma(a) = m_1 a + m_2$ will be treated by a very different method in a posterior paper by the author. The techniques unfortunately do not extend to the remaining cases, even for the seemingly simple case when $\gamma(a) = 1/a$.

1.2. A double sum over squarefree integers. Developping the squares in $\mathcal{M}_2[m](X, q)$, we obtain the equality

$$(1.14) \quad \mathcal{M}_2[m](X, q) = S[m](X, q) - 2C(q) \frac{X}{q} \sum_{\substack{n \leq X \\ (n, q)=1}} \mu^2(n) + \varphi(q) \left(C(q) \frac{X}{q} \right)^2,$$

where

$$(1.15) \quad C(q) = \frac{6}{\pi^2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^{-1},$$

and $S[m](X, q)$ is the double sum

$$(1.16) \quad S[m](X, q) = \sum_{\substack{n_1, n_2 \leq X \\ (n_1 n_2, q)=1 \\ mn_1 = n_2 \pmod{q}}} \mu^2(n_1) \mu^2(n_2).$$

As in the classical dispersion method, it is important to obtain an asymptotic formula for each of the terms in (1.14), for $S[m](X, q)$ in particular. We point out that since the order of magnitude of $\mathcal{M}_2[m](X, q)$ is much smaller than some of the terms in the sum (1.14), our asymptotic formula for $S[m](X, q)$ must be precise enough to produce huge cancellations between the main terms. This precision is contained in

Theorem 1.4. *Let $X > 2$ be a real number and let m be a nonzero integer of arbitrary sign. Then for every q integer, satisfying $q \leq X$, we have*

$$(1.17) \quad S[m](X, q) = \frac{\varphi(q)}{2} \left(\frac{C(q)X}{q} \right)^2 + \frac{C}{2} \Gamma_{\text{an}}(m) \Gamma_{\text{ar}}(m) \prod_{p|q} \left(1 + 2p^{-1} \right)^{-1} X^{1/2} q^{1/2} \\ + O_m \left(d(q) X^{1/3} q^{2/3} + X^{23/15} q^{-13/15} (\log X)^{15} \right),$$

where $C, C(q)$ are as in (1.8) and (1.15) respectively, $\Gamma_{\text{an}}(m)$ and $\Gamma_{\text{ar}}(m)$ are as in (1.11) and (1.12) respectively; and the implied $O_{\epsilon, m}$ -constant depends at most on ϵ and m .

Some parts of the proof of Theorem 1.4 were inspired by the paper of Heath-Brown [6] where he proved (see [6, Theorem 2]) that for every $X > 2$, one has

$$(1.18) \quad \sum_{n < X} \mu^2(n) \mu^2(n+1) = C_2 X + O(X^{7/11} (\log X)^7),$$

where C_2 is given by

$$(1.19) \quad C_2 = \prod_p \left(1 - 2p^{-2} \right) = 0.322\,634\,098 \dots$$

For an improve on formula (1.18), see Reuss [10]. The proof of (1.17) will cover the main part of this article and the Theorem 1.2 will follow.

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2. NOTATION

For $X > 1$ we write $\mathcal{L} := \log 2X$

Let ω be the well-known function defined by

$$(2.1) \quad \omega(n) = \sum_{p|n} 1,$$

for every integer $n \geq 1$. For $N > 0$, we use the notation $n \sim N$ meaning $N < n \leq 2N$.

Let $\psi(v)$ be the sawtooth function defined, as in Titchmarsh [13], by

$$(2.2) \quad [v] = v - \frac{1}{2} + \psi(v).$$

For $n \in \mathbb{Z}, n \neq 0$ we define

$$(2.3) \quad \sigma(n) = \prod_{p^2|n} p.$$

3. INITIAL STEPS

We are considering

$$S[m](X, q) = \sum_{\substack{n_1, n_2 \leq X \\ (n_1 n_2, q) = 1 \\ mn_1 = n_2 \pmod{q}}} \mu^2(n_1) \mu^2(n_2).$$

We write $\ell q = n_2 - mn_1$ and write $S[m](X, q)$ as a sum over ℓ . We have

$$(3.1) \quad S[m](X, q) = \sum_{\ell \in \mathbb{Z}} \sum_{\substack{n \in I(\ell) \\ (n, q) = 1}} \mu^2(n) \mu^2(mn + \ell q),$$

where $I(\ell) = I(\ell; m, q)$ is the interval defined by

$$(3.2) \quad I(\ell) = \begin{cases} (0, X) \cap \left(\frac{-\ell q}{m}, \frac{X - \ell q}{m} \right) & \text{if } m > 0 \\ (0, X) \cap \left(\frac{X - \ell q}{m}, \frac{-\ell q}{m} \right) & \text{if } m < 0. \end{cases}$$

That is,

$$I(\ell) = \{n; n \text{ and } mn + \ell q \in (0, X)\}.$$

Remark 3.1. Observe that since $\mu^2(n) \mu^2(mn + h) = 1$ if and only if $\sigma(n) \sigma(mn + h) = 1$ (recall definition (2.3) of $\sigma(n)$), we have

$$(3.3) \quad \mu^2(n) \mu^2(mn + h) = \sum_{d | \sigma(n) \sigma(mn + h)} \mu(d).$$

We now replace formula (3.3) with $h = \ell q$ in equation (3.1) and invert order of summation. We thus obtain

$$(3.4) \quad S[m](X, q) = \sum_{\ell \in \mathbb{Z}} \sum_{\substack{1 \leq d \leq X \\ (d, q) = 1}} \mu(d) N_d(\ell),$$

where

$$(3.5) \quad N_d(\ell) := \#\{n \in I(\ell); (n, q) = 1 \text{ and } \sigma(n) \sigma(mn + \ell q) \equiv 0 \pmod{d}\}.$$

Let $0 < y < X$ be a parameter to be chosen later depending on X and q . We break down the sum (3.4) as follows

$$(3.6) \quad S[m](X, q) = S_{\leq y}[m](X, q) + S_{> y}[m](X, q),$$

where,

$$(3.7) \quad \begin{cases} S_{\leq y}[m](X, q) = \sum_{\ell \in \mathbb{Z}} \sum_{\substack{d \leq y \\ (d, q)=1}} \mu(d) N_d(\ell), \\ S_{> y}[m](X, q) = \sum_{\ell \in \mathbb{Z}} \sum_{\substack{y < d \leq X \\ (d, q)=1}} \mu(d) N_d(\ell). \end{cases}$$

From this point on, we make the hypothesis

m is squarefree.

For the general proof one would need to distinguish two different cases in the definition of the function κ (see (3.10) below). In the next lemma we study the first of the sums on the right-hand side of (3.6).

Lemma 3.1. *Let $m \neq 0$ be a squarefree number and $y > 1$. Let $S_{\leq y}[m](X, q)$ be defined by (3.7). Then for every $y > 1$, we have*

$$(3.8) \quad S_{\leq y}[m](X, q) = \sum_{\ell \in \mathbb{Z}} f_q(\ell, m) |I(\ell)| + O_m \left(\frac{X}{q} (\tau(q)y + Xy^{-1}) \log y \right),$$

uniformly for $X > 1$ and $q \geq 1$ satisfying $(m, q) = 1$, where

$$(3.9) \quad f_q(\ell, m) = \prod_p \left(1 - \frac{2}{p^2} \right) \prod_{p|m} \left(\frac{p^2 - 1}{p^2 - 2} \right) \prod_{p|q} \left(\frac{p^2 - p}{p^2 - 2} \right) \kappa((\ell, m^2)) \prod_{\substack{p^2 | \ell \\ p \nmid mq}} \left(\frac{p^2 - 1}{p^2 - 2} \right),$$

κ is the multiplicative function such that

$$(3.10) \quad \kappa(p^\alpha) = \begin{cases} \frac{p^2 - p - 1}{p^2 - 1}, & \text{if } \alpha = 1, \\ \frac{p^2 - p}{p^2 - 1}, & \text{if } \alpha = 2, \\ 0, & \text{if } \alpha \geq 3, \end{cases}$$

and the implied constant depends at most on m .

Proof. We want to evaluate $S_{\leq y}[m](X, q)$. Let $p \nmid q$, we define

$$(3.11) \quad u_p(\ell) = \#\{v \pmod{p^2}; v \equiv 0 \pmod{p^2} \text{ or } mv + \ell q \equiv 0 \pmod{p^2}\}.$$

Again, we omit the dependence in m and q since both are held fixed and no confusion should arise from this. By the Chinese Remainder Theorem, we have, for every d squarefree, $(d, q) = 1$.

$$(3.12) \quad N_d(\ell) = \frac{\varphi(q)}{q} U_d(\ell) \frac{|I(\ell)|}{d^2} + O(d(q) U_d(\ell)),$$

where

$$U_d(\ell) := \prod_{p|d} u_p(\ell).$$

We notice further that if $(p, m) = 1$, $u_p \leq 2$. Hence we have the upper bound

$$(3.13) \quad U_d(\ell) \ll_m 2^{\omega(d)}.$$

We use formula (3.12) and the upper bound (3.13). So, we deduce

$$(3.14) \quad \begin{aligned} \sum_{\substack{d \leq y \\ (d, q)=1}} \mu(d) N_d(\ell) &= \sum_{\substack{d \leq y \\ (d, q)=1}} \mu(d) \frac{\varphi(q)}{q} U_d(\ell) \frac{|I(\ell)|}{d^2} + O_m(d(q)y \log y) \\ &= \frac{\varphi(q)}{q} \prod_{p \nmid q} \left(1 - \frac{u_p(\ell)}{p^2} \right) |I(\ell)| + O_m(\tau(q)y \log y + Xy^{-1} \log y), \end{aligned}$$

where in the second line we used that $|I(\ell)| \leq X$ and the convergence of the appearing infinite product. We proceed by realizing that if $|\ell| > \frac{(|m|+1)X}{q}$, the set $I(\ell)$ is empty and hence $N_d(\ell) = 0$ for every d . This observation and equation (3.14) combined give us

$$\begin{aligned}
 S_{\leq y}[m](X, q) &= \sum_{|\ell| \leq \frac{(|m|+1)X}{q}} \sum_{\substack{d \leq y \\ (d, q)=1}} \mu(d) N_d(\ell) \\
 &= \frac{\varphi(q)}{q} \sum_{|\ell| \leq \frac{(|m|+1)X}{q}} \prod_{p \nmid q} \left(1 - \frac{u_p(\ell)}{p^2}\right) |I(\ell)| + O_m \left(\frac{X}{q} (\tau(q)y + Xy^{-1}) \log y \right) \\
 (3.15) \quad &= \frac{\varphi(q)}{q} \sum_{\ell \in \mathbb{Z}} \prod_{p \nmid q} \left(1 - \frac{u_p(\ell)}{p^2}\right) |I(\ell)| + O_m \left(\frac{X}{q} (\tau(q)y + Xy^{-1}) \log y \right).
 \end{aligned}$$

We finish by a study of $u_p(\ell)$ for every $p \nmid q$. We distinguish five different cases.

- If $p \mid m, p^2 \mid \ell$ then

$$u_p(\ell) = p,$$

- If $p \mid m, p \mid \ell$ but $p^2 \nmid \ell$ then

$$u_p(\ell) = p + 1,$$

- If $p \mid m, p \nmid \ell$ then

$$u_p(\ell) = 1,$$

- If $p \nmid m, p^2 \mid \ell$ then

$$u_p(\ell) = 1,$$

- If $p \nmid m, p^2 \nmid \ell$ then

$$u_p(\ell) = 2.$$

The lemma is now a consequence of (3.15) and the different values of $u_p(\ell)$. \square

Now, we can go back to the formula (3.6) and use (3.8) for $S_{\leq y}[m](X, q)$. What we obtain is the following

$$(3.16) \quad S[m](X, q) = \mathcal{A}[m](X, q) + S_{> y}[m](X, q) + O_m \left(\frac{X}{q} (\tau(q)y + Xy^{-1}) \log y \right),$$

where

$$(3.17) \quad \mathcal{A}[m](X, q) = \sum_{\ell \in \mathbb{Z}} f_q(\ell, m) |I(\ell)|.$$

We finish this section by introducing the following multiplicative function for which we will deduce some properties in the next section. We let

$$(3.18) \quad h(d) := \mu^2(d) \prod_{p \mid d} (1 - 2p^{-2})^{-1}.$$

We point out further that we have the following equality

$$(3.19) \quad \sum_{\substack{d^2 \mid \ell \\ (d, r)=1}} \frac{h(d)}{d^2} = \prod_{\substack{p^2 \mid \ell \\ p \nmid r}} \left(\frac{p^2 - 1}{p^2 - 2} \right),$$

for every $\ell, r \in \mathbb{Z}$, both nonzero.

4. PREPARATION RESULTS

In the next section we evaluate the sum $\mathcal{A}[m](X, q)$. But first we prove some preliminary results that shall be useful.

We start with a lemma that is a simplified version of [3, Lemma 1].

Lemma 4.1. *For $X > 1$, $0 < s < 2$, and $\psi(v)$ defined by (2.2),*

$$\int_0^X \psi(v) v^{-s/2} dv = \frac{\zeta(\frac{s}{2} - 1)}{(\frac{s}{2} - 1)} + O(X^{-\frac{s}{2}}),$$

where the O -constant is absolute.

Proof. We start with the formula (see [13, equation (2.1.6)]),

$$\zeta\left(\frac{s}{2} - 1\right) = \left(\frac{s}{2} - 1\right) \int_0^\infty \psi(v) v^{-s/2} dv, \quad (0 < s < 2).$$

We estimate the tail of the integral

$$(4.1) \quad I = \int_X^\infty \psi(v) v^{-s/2} dv.$$

Let $\Psi_1(v)$ be a primitive integral of $\psi(v)$ which satisfies $\Psi_1(0) = 0$. So, integration by parts gives us

$$I = \Psi_1(X) X^{-\frac{s}{2}} + \frac{s}{2} \int_X^\infty \psi_1(v) v^{-\frac{s}{2}-1} dv.$$

Notice that since $\int_0^1 \psi(v) dv = 0$, Ψ_1 is periodic and thus bounded. Hence, we have

$$|I| \leq 2X^{-\frac{s}{2}}$$

which concludes the proof. \square

The next lemma writes the function $h(d)$ as a convolution between the identity and a function which decays rapidly.

Lemma 4.2. *Let $h(d)$ be as in (3.18). Then we have*

$$h(d) = \sum_{d_1 d_2 = d} \beta(d_1),$$

where $\beta(t)$ is supported on the cubefree numbers. Furthermore, for every cubefree number t , if we write $t = ab^2$ with a, b squarefree, $(a, b) = 1$, we have

$$\beta(t) \ll \frac{d(a)}{a^2}.$$

Proof. It is easy to calculate the values of β on prime powers

$$\beta(p^k) = \begin{cases} 1, & \text{if } k = 0, \\ \frac{2}{p^2-2}, & \text{if } k = 1, \\ \frac{-p}{p^2-2}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

The lemma now follows by noticing that the product

$$\prod_p \left(\frac{p^2}{p^2-2} \right)$$

is convergent. \square

The lemma that comes next is based on [3, Lemma 3]. However, there are some confusing steps in the corresponding proof as it was also pointed out by Vaughan (see [15, page 574]). In particular, the value for the constant B (see [3, equation 6.3]) is inaccurate wrong. In a previous version of this paper we proceeded by a modification of Croft's arguments which led to basically the same result as stated in [3, Lemma 3]. More recently, inspired by a paper by Pillichshammer [9], we found a way of obtaining a slightly better result by elementary means. The author later became aware of the paper [5], where it is implicitly contained a similar lemma with $Y^{1/3}$ replaced by $Y^{1/3+\epsilon}$ that uses again methods from complex analysis.

Lemma 4.3. *(Compare to [3, Lemma 3]) For $Y > 0$ and r an integer ≥ 1 , let*

$$G(Y, r) := \sum_{(d, r)=1} h(d) \int_0^{Y/d^2} \psi(v) dv,$$

where $\psi(v)$ is defined by (2.2) and $h(d)$ by (3.18). We have, uniformly for $Y \geq 1$ and $r \neq 0$,

$$(4.2) \quad \sum_{(d, r)=1} \int_0^{Y/d^2} \psi(v) dv = \frac{\varphi(r)}{r} \frac{\zeta(3/2)}{2\pi} Y^{1/2} + O\left(d(r)Y^{1/3}\right),$$

and,

$$(4.3) \quad G(Y, r) = C' \prod_{p|r} (1 + p(p^2 - 2)^{-1})^{-1} Y^{1/2} + O\left(d(r)Y^{1/3}\right),$$

where C' is given by

$$(4.4) \quad C' = \frac{\zeta(\frac{3}{2})}{2\pi} \prod_p \left(\frac{p^3 - 3p + 2}{p(p^2 - 2)} \right),$$

and the O -constant is absolute.

Proof. We start by proving formula (4.2) which is simpler. Let $D = D(Y) > 0$ to be chosen later. Since $\psi(v)$ is periodic, we have

$$(4.5) \quad \begin{aligned} \sum_{(d, r)=1} \int_0^{Y/d^2} \psi(v) dv &= \sum_{\substack{d \leq D \\ (d, r)=1}} \int_0^{Y/d^2} \psi(v) dv + \sum_{\substack{d > D \\ (d, r)=1}} \int_0^{Y/d^2} \psi(v) dv \\ &= \sum_{\substack{d > D \\ (d, r)=1}} \int_0^{Y/d^2} \psi(v) dv + O(D). \end{aligned}$$

By inverting order of the summation and the integration, we have

$$\begin{aligned} \sum_{\substack{d > D \\ (d, r)=1}} \int_0^{Y/d^2} \psi(v) dv &= \int_0^{Y/D^2} \psi(v) \sum_{\substack{D < d \leq \sqrt{\frac{Y}{v}} \\ (d, r)=1}} 1 dv \\ &= \int_0^{Y/D^2} \psi(v) \left(\frac{\varphi(r)}{r} ((Y/v)^{1/2} - D) + O(d(r)) \right) dv \\ &= \frac{\varphi(r)}{r} Y^{1/2} \int_0^{Y/D^2} \psi(v) v^{-1/2} dv + O(D) + O\left(d(r) \frac{Y}{D^2}\right). \end{aligned}$$

By the lemma 4.1, the equation above implies

$$(4.6) \quad \sum_{\substack{d > D \\ (d,r)=1}} \int_0^{Y/d^2} \psi(v) dv = \frac{-2\varphi(r)}{r} \zeta\left(-\frac{1}{2}\right) Y^{1/2} + O(D) + O\left(d(r) \frac{Y}{D^2}\right).$$

We make the choice $D = Y^{1/3}$. Equation (4.2) is now just a consequence of (4.5), (4.6) and the functional equation for the Riemann zeta function, which for $s = -1/2$ gives

$$\zeta(-1/2) = -\frac{\zeta(3/2)}{4\pi}.$$

We will now deduce (4.3) from (4.2).
Since

$$h(d) = (1 * \beta)(d),$$

we can write

$$(4.7) \quad G(Y, r) = \sum_{(d_1, r)=1} \beta(d_1) \sum_{(d_2, r)=1} \int_0^{Y/d_1^2 d_2^2} \psi(v) dv.$$

We have two possibilities. If $d_1^2 \leq Y$, we shall use formula (4.2) for the inner sum on the right-hand side of (4.7). If, otherwise, $d_1^2 > Y$, we shall use the trivial bound

$$\sum_{(d_2, r)=1} \int_0^{Y/d_1^2 d_2^2} \psi(v) dv \leq \sum_{d_2} \frac{Y}{d_1^2 d_2^2} \ll \frac{Y}{d_1^2}.$$

By doing so in formula (4.7), we obtain

$$(4.8) \quad G(Y, r) = \sum_{\substack{d_1 \leq \sqrt{Y} \\ (d_1, r)=1}} \beta(d_1) \frac{\varphi(r)}{r} \frac{\zeta(3/2)}{2\pi} \left(\frac{Y}{d_1^2}\right)^{1/2} \\ + O\left(d(r) \sum_{\substack{d_1 \leq \sqrt{Y} \\ (d_1, r)=1}} \beta(d_1) \left(\frac{Y}{d_1^2}\right)^{1/3} + \sum_{\substack{d_1 > \sqrt{Y} \\ (d_1, r)=1}} \beta(d_1) \frac{Y}{d_1^2}\right).$$

By completing the first term and using lemma 4.2, we have

$$(4.9) \quad G(Y, r) = C_\beta(r) \frac{\varphi(r)}{r} \frac{\zeta(3/2)}{2\pi} Y^{1/2} + O\left(d(r) Y^{1/3} \sum_{ab^2 \leq \sqrt{Y}} \sum \frac{d(a)}{a^{8/3} b^{4/3}}\right) \\ + O\left(Y^{1/2} \sum_{ab^2 > \sqrt{Y}} \sum \frac{d(a)}{a^3 b^2} + Y \sum_{ab^2 > \sqrt{Y}} \sum \frac{d(a)}{a^4 b^4}\right),$$

where

$$C_\beta(r) = \sum_{(d_1, r)=1} \frac{\beta(d_1)}{d_1} = \prod_p \left(1 - \frac{p-2}{p(p^2-2)}\right) \prod_{p|r} \left(1 - \frac{p-2}{p(p^2-2)}\right)^{-1}.$$

The first error term on the right-hand side of (4.9) is clearly $\ll d(r)Y^{1/3}$. For the second one, we have

$$\begin{aligned} \sum_{ab^2 > \sqrt{Y}} \frac{d(a)}{a^3 b^2} &\leq \sum_{ab > Y^{1/4}} \frac{d(a)}{a^2 b^2} \\ &\ll Y^{-1/4} (\log Y)^2, \end{aligned}$$

and analogously,

$$\sum_{ab^2 > \sqrt{Y}} \frac{d(a)}{a^4 b^4} \ll Y^{-3/4} (\log Y)^2.$$

Once we inject the above upper bounds in equation (4.9), we obtain

$$G(Y, r) = C' \prod_{p|r} (1 + p(p^2 - 2)^{-1})^{-1} Y^{1/2} + O\left(d(r)Y^{1/3} + Y^{1/4}(\log Y)^2\right),$$

which concludes the proof of formula (4.3). \square

In the following lemma we gather a series of identities that shall be useful later on, and whose proofs are routine and, hence, omitted.

Lemma 4.4. *Let m be a squarefree integer and let $\kappa(\rho)$ be as in (3.10), then we have the equalities*

$$\begin{cases} \sum_{\rho \cdot \sigma | m^2} \frac{\kappa(\rho)\mu(\sigma)}{\rho\sigma} = \prod_{p|m} \left(\frac{p^2 - 1}{p^2} \right), \\ \sum_{\rho \cdot \sigma | m^2} \kappa(\rho)\mu(\sigma) = \prod_{p|m} \left(\frac{p^2 - p}{p^2 - 1} \right), \\ \sum_{\rho \cdot \sigma | m^2} \kappa(\rho)\mu(\sigma)\rho^{1/2}\sigma^{1/2} = \prod_{p|m} \left(\frac{p^2 - p^{3/2} + p - 1}{p^2 - 1} \right). \end{cases}$$

Also, let $r \neq 0$ be an integer, and let $h(d)$ be as in (3.18). Then we have further the equalities

$$\begin{cases} \sum_{(d,r)=1} \frac{h(d)}{d^4} = \prod_p \left(\frac{(p^2 - 1)^2}{p^2(p^2 - 2)} \right) \prod_{p|r} \left(\frac{(p^2 - 1)^2}{p^2(p^2 - 2)} \right)^{-1}, \\ \sum_{(d,r)=1} \frac{h(d)}{d^2} = \prod_p \left(\frac{p^2 - 1}{p^2 - 2} \right) \prod_{p|r} \left(\frac{p^2 - 1}{p^2 - 2} \right)^{-1}. \end{cases}$$

Finally, let $m, q \in \mathbb{Z}$, $m > 0$ squarefree and $q \neq 0$. Also let $f_q(\ell, m)$ be as in (3.9) and $C(q)$ be as in definition (1.15). Then, we have the following equality

$$f_q(0, m) = \frac{\varphi(mq)}{mq} C(mq)$$

We let

$$(4.10) \quad \mathfrak{S}[m](Y, q) := \sum_{0 < \ell \leq Y} f_q(\ell, m)(Y - \ell).$$

The following lemma gives a formula for $\mathfrak{S}[m](Y, q)$ by means of the previous lemmas.

Proposition 4.5. *Let $m \neq 0$ be an integer and let $f_q(\ell, m)$ be defined by (3.9). Then, uniformly for $Y > 0$ and q positive integer, we have*

$$(4.11) \quad \begin{aligned} \mathfrak{S}[m](Y, q) &= \frac{\varphi(q)}{q} C(q)^2 Y^2 - \frac{\varphi(mq)}{mq} C(mq) Y \\ &\quad + \frac{C}{2} \Gamma_{\text{ar}}(m) \prod_{p|q} (1 + 2p^{-1})^{-1} Y^{1/2} + O_m(d(q)Y^{1/3}), \end{aligned}$$

where $C, \Gamma_{\text{ar}}(m)$ are as in (1.8), (1.12) respectively; $C(r)$ is as in (1.15) for $r = q, mq$ and the implied O_m -constant depends at most on m .

Proof. We start by recalling (3.9) and the formula (3.19) for last product. We have

$$f_q(\ell, m) = C_2 \prod_{p|m} \left(\frac{p^2 - 1}{p^2 - 2} \right) \prod_{p|q} \left(\frac{p^2 - p}{p^2 - 2} \right) \kappa((\ell, m^2)) \sum_{\substack{d^2|\ell \\ (d, mq)=1}} \frac{h(d)}{d^2},$$

where C_2 is as in (1.19). We notice that the three first terms on the right-hand side of equation above are independent of ℓ that means that in order to evaluate $\mathfrak{S}[m](X, q)$, we need to study

$$\begin{aligned} (4.12) \quad \mathfrak{S}'[m](Y, q) &:= \sum_{0 < \ell \leq Y} \kappa((\ell, m^2)) \sum_{\substack{d^2|\ell \\ (d, mq)=1}} \frac{h(d)}{d^2} (Y - \ell) \\ &= \sum_{\rho|m^2} \kappa(\rho) \sum_{\substack{0 < \ell \leq Y \\ (\ell, m^2)=\rho}} (Y - \ell) \sum_{\substack{d^2|\ell \\ (d, mq)=1}} \frac{h(d)}{d^2} \\ (4.13) \quad &= \sum_{\rho\sigma|m^2} \kappa(\rho) \mu(\sigma) \rho \sigma \sum_{(d, mq)=1} \frac{h(d)}{d^2} \sum_{\substack{0 < \ell_0 \leq \frac{Y}{\rho\sigma} \\ d^2|\ell_0}} \left(\frac{Y}{\rho\sigma} - \ell_0 \right) \end{aligned}$$

where in the third line we used Möbius inversion formula for detecting the gcd condition. We proceed to write the inner sum as an integral. We have

$$\begin{aligned} \sum_{\substack{0 < \ell_0 \leq \frac{Y}{\rho\sigma} \\ d^2|\ell_0}} \left(\frac{Y}{\rho\sigma} - \ell_0 \right) &= \sum_{\substack{0 < \ell_0 \leq \frac{Y}{\rho\sigma} \\ d^2|\ell_0}} \int_{\ell}^{\frac{Y}{\rho\sigma}} 1 du \\ (4.14) \quad &= \int_0^{\frac{Y}{\rho\sigma}} \sum_{\substack{0 < \ell_0 \leq u \\ d^2|\ell_0}} 1 du = \int_0^{\frac{Y}{\rho\sigma}} \left\lfloor \frac{u}{d^2} \right\rfloor du. \end{aligned}$$

Recall formula (2.2) defining the sawtooth function ψ

$$\lfloor x \rfloor = x - \frac{1}{2} + \psi(x).$$

This formula when used in equation (4.14) gives

$$\sum_{\substack{0 < \ell_0 \leq \frac{Y}{\rho\sigma} \\ d^2|\ell_0}} \left(\frac{Y}{\rho\sigma} - \ell_0 \right) = \frac{Y^2}{2\rho^2\sigma^2d^2} - \frac{Y}{2\rho\sigma} + d^2 \int_0^{\frac{Y}{\rho\sigma d^2}} \psi(v) dv,$$

where, here, we calculated the integrals for the first two terms and made the change of variables $v = u/d^2$ on the last one. Injecting this formula in (4.12), we deduce

$$(4.15) \quad \mathfrak{S}'[m](Y, q) = \lambda_2(m, q) Y^2 - \lambda_1(m, q) Y + \sum_{\rho\sigma|m^2} \kappa(\rho) \mu(\sigma) \rho \sigma G\left(\frac{Y}{\rho\sigma}, mq\right),$$

where

$$\lambda_2(m, q) = \sum_{\rho\sigma|m^2} \frac{\kappa(\rho) \mu(\sigma)}{\rho\sigma} \times \sum_{(d, mq)=1} \frac{h(d)}{d^4}$$

and

$$\lambda_1(m, q) = \sum_{\rho\sigma|m^2} \kappa(\rho)\mu(\sigma) \times \sum_{(d, mq)=1} \frac{h(d)}{d^2}$$

As for $G\left(\frac{Y}{\rho\sigma}, mq\right)$, there are two possibilities. Either $\frac{Y}{\rho\sigma} \geq 1$ and we can use formula (4.3) giving

$$G\left(\frac{Y}{\rho\sigma}, mq\right) = C' \prod_{p|mq} (1 + p(p^2 - 2)^{-1})^{-1} \left(\frac{Y}{\rho\sigma}\right)^{1/2} + O\left(d(mq) \left(\frac{Y}{\rho\sigma}\right)^{1/3}\right).$$

On the other hand, if $\frac{Y}{\rho\sigma} < 1$, we have both $\left(\frac{Y}{\rho\sigma}\right)^{1/2} \ll 1$ and

$$G\left(\frac{Y}{\rho\sigma}, mq\right) \ll \sum_d \frac{Y}{\rho\sigma d^2} \ll 1.$$

Hence we can write, in this case,

$$G\left(\frac{Y}{\rho\sigma}, mq\right) = C' \prod_{p|mq} (1 + p(p^2 - 2)^{-1})^{-1} \left(\frac{Y}{\rho\sigma}\right)^{1/2} + O(1).$$

The above considerations now add together giving

$$(4.16) \quad \sum_{\rho\sigma|m^2} \kappa(\rho)\mu(\sigma)\rho\sigma G\left(\frac{Y}{\rho\sigma}, mq\right) = \lambda(m, q)Y^{1/2} + O_m(d(q)Y^{1/3}),$$

where

$$\lambda(m, q) = C' \prod_{p|mq} \left(1 + \frac{p}{p^2 - 2}\right)^{-1} \sum_{\rho\sigma|m^2} \kappa(\rho)\mu(\sigma)\rho^{1/2}\sigma^{1/2}.$$

Now, if we go back to equation (4.15) and use (4.16) above we obtain

$$\mathfrak{S}'[m](Y, q) = \lambda_2(m, q)Y^2 - \lambda_1(m, q)Y + \lambda(m, q)Y^{1/2} + O_m(d(q)Y^{1/3}).$$

Hence

$$(4.17) \quad \mathfrak{S}[m](Y, q) = \Lambda_2(m, q)Y^2 - \Lambda_1(m, q)Y + \Lambda(m, q)Y^{1/2} + O_m(d(q)Y^{1/3}),$$

where

$$\begin{cases} \Lambda(m, q) = C_2 \prod_{p|m} \left(\frac{p^2 - 1}{p^2 - 2}\right) \prod_{p|q} \left(\frac{p^2 - p}{p^2 - 2}\right) \lambda(m, q), \\ \Lambda_i(m, q) = C_2 \prod_{p|m} \left(\frac{p^2 - 1}{p^2 - 2}\right) \prod_{p|q} \left(\frac{p^2 - p}{p^2 - 2}\right) \lambda_i(m, q), \quad i = 1, 2. \end{cases}$$

Lemma 4.4 ensures that the constants $\Lambda_2(m, q)$, $\Lambda_1(m, q)$ and $\Lambda(m, q)$ correspond to the constants in (4.11). The result now follows from equation (4.17). \square

5. MAIN TERM

In this section we show how to use the results from the previous section to exhibit a formula for $\mathcal{A}[m](X, q)$, where the second main term in formula (1.17) makes its first appearance. We prove the following

Proposition 5.1. *For $X > 1$, m, q integers such that m is squarefree and $q \geq 1$, let $\mathcal{A}[m](X, q)$ be defined by formula (3.17). Then we have, uniformly for $X, q > 1$,*

$$(5.1) \quad \mathcal{A}[m](X, q) = \varphi(q) \left(C(q) \frac{X}{q}\right)^2 + \frac{C}{2} \Gamma_{\text{an}}(m) \Gamma_{\text{ar}}(m) \prod_{p|q} (1 + 2p^{-1})^{-1} X^{1/2} q^{1/2} \\ + O_m\left(d(q) X^{1/3} q^{2/3}\right),$$

where C , $C(r)$, $\Gamma_{\text{an}}(m)$ and $\Gamma_{\text{ar}}(m)$ are as in (1.8), (1.15), (1.11) and (1.12) respectively; and the implied O_m -constant depends at most on m .

Proof. There is a slighty difference depending on whether $m > 0$ or $m < 0$. So we study the two cases separately.

The case $m > 0$. We start from the formula

$$\mathcal{A}[m](X, q) = \sum_{\ell \in \mathbb{Z}} f_q(\ell, m) |I(\ell)|.$$

By analyzing the possible values of $|I(\ell)|$, we have

$$(5.2) \quad \mathcal{A}[m](X, q) = f_q(0, m) \frac{X}{m} + \sum_{0 < \ell \leq \frac{X}{q}} f_q(\ell, m) \left(\frac{X - \ell q}{m} \right) \\ + \sum_{-\frac{(m-1)X}{q} \leq \ell < 0} f_q(\ell, m) \frac{X}{m} + \sum_{-\frac{mX}{q} \leq \ell < -\frac{(m-1)X}{q}} f_q(\ell, m) \left(\frac{X + \ell q}{m} \right).$$

We remark that for m, q fixed, $f_q(\ell, m)$ only depends on the positive divisors of ℓ (see formula (3.9)). Hence $f_q(\ell, m) = f_q(-\ell, m)$. As a consequence, formula (5.2) implies

$$\mathcal{A}[m](X, q) = f_q(0, m) \frac{X}{m} + \frac{1}{m} \sum_{0 < \ell \leq \frac{X}{q}} f_q(\ell, m) (X - \ell q) \\ - \frac{1}{m} \sum_{0 < \ell \leq \frac{(m-1)X}{q}} f_q(\ell, m) ((m-1)X - \ell q) + \frac{1}{m} \sum_{0 < \ell \leq \frac{mX}{q}} f_q(\ell, m) (mX - \ell q).$$

We recall that $\mathfrak{S}[m](Y, q) = \sum_{0 < \ell \leq Y} f_q(\ell, m) (Y - \ell)$. So, we can write

$$(5.3) \quad \mathcal{A}[m](X, q) = f_q(0, m) \frac{X}{m} + \frac{q}{m} \left\{ \mathfrak{S}[m](X/q, q) - \mathfrak{S}[m]((m-1)X/q, q) + \mathfrak{S}[m](mX/q, q) \right\}.$$

By proposition 4.5, we have

$$\mathfrak{S}[m](Y, q) = \frac{\varphi(q)}{q} C(q)^2 Y^2 + \frac{\varphi(mq)}{mq} C(mq) Y + \frac{C}{2} \Gamma_{\text{ar}}(m) \prod_{p|q} (1 + 2p^{-1})^{-1} Y^{1/2} + O_m \left(d(q) Y^{1/3} \right).$$

Hence, we deduce from (5.3) that

$$\mathcal{A}[m](X, q) = f_q(0, m) \frac{X}{m} + \varphi(q) C(q)^2 \left(\frac{X}{q} \right)^2 - \frac{\varphi(mq)}{mq} C(mq) \frac{X}{m} \\ + \frac{C}{2} \Gamma_{\text{an}}(m) \Gamma_{\text{ar}}(m) \prod_{p|q} (1 + 2p^{-1})^{-1} X^{1/2} q^{1/2} + O_m \left(d(q) X^{1/3} q^{2/3} \right),$$

where $\Gamma_{\text{an}}(m)$ is as in (1.11). Now, by lemma 4.4, the first and third terms disappear. This concludes the proof of formula (5.1) in this case.

The case $m < 0$. Analogously to the previous case, we have

$$\mathcal{A}[m](X, q) = \frac{q}{m} \{ \mathfrak{S}[m](X/q, q) + \mathfrak{S}[m](-mX/q, q) - \mathfrak{S}[m]((1-m)X/q, q) \}$$

and, again by proposition 4.5, we have that (5.1) is true in this case as well. The proof of the proposition is finished. \square

6. BOUNDING $S_{>y}[m](X, q)$

In the present section we give a bound for $S_{>y}[m](X, q)$. We start by noticing that $d^2 \mid \xi(n; \ell q, m)$ if and only if there exist j, k such that $d = jk$ and both $j^2 \mid n$ and $k^2 \mid mn + \ell q$. Moreover since we are supposing $n, mn + \ell q < X$, we have $j, k < \sqrt{X}$. From this observation we deduce

$$\begin{aligned} S_{>y}[m](X, q) &= \sum_{\ell \in \mathbb{Z}} \sum_{\substack{y < d \leq X \\ (d, q) = 1}} \mu(d) \# \{ n \in I(\ell); (n, q) = 1 \text{ and } \xi(n; \ell q, m) \equiv 0 \pmod{d} \} \\ &\leq \sum_{\substack{j, k \leq \sqrt{X} \\ jk > y \\ (jk, q) = 1}} \# \{ (n, \ell) \in \mathbb{Z}^2; 0 < n, mn + \ell q < X \text{ and } j^2 \mid n, k^2 \mid mn + \ell q \} \\ (6.1) \quad &= \sum_{\substack{j, k \leq \sqrt{X} \\ jk > y \\ (jk, q) = 1}} N[m](X, q; j, k), \end{aligned}$$

by definition.

We shall divide the possible values of j and k into sets of the form

$$\mathcal{B}(J, K) := \{ (j, k); \gcd(jk, q) = 1, j \sim J, k \sim K \}.$$

We can do the division using at most $O(\mathcal{L}^2)$ since we are summing over $j, k \leq X^{1/2}$.

Let

$$\begin{aligned} \mathcal{N}[m](J, K) &= \sum_{\substack{(jk, q) = 1 \\ j \sim J, k \sim K}} N[m](X, q; j, k) \\ (6.2) \quad &= \# \{ (j, k, u, v); j \sim J, k \sim K, 0 < j^2 u, k^2 v < X, \text{ and } mj^2 u \equiv k^2 v \pmod{q} \} \end{aligned}$$

By taking the maximum over all J, K , we obtain a pair (J, K) with $J, K \leq X^{1/2}$ such that

$$(6.3) \quad S_{>y}[m](X, q) \ll \mathcal{N}[m](J, K) \mathcal{L}^2.$$

By the condition

$$jk > y$$

in formula (6.1), we can also impose

$$JK \geq \frac{y}{4}.$$

Our problem is now bounding $\mathcal{N}[m](J, K)$. Notice that, since we bound $S_{>y}[m](X, q)$ as in (6.3), we will not be able to benefit from oscillations of the coefficients $\mu(d)$ in the definition (3.7). Although formula (6.2) is not symmetrical repectively to J and K , we would like to benefit from some symmetry. With that in mind, let $m_1, m_2 \in \mathbb{Z}$ we define

$$(6.4) \quad \mathcal{N}[m_1, m_2](J, K) := \# \{ (j, k, u, v); j, k \in \mathcal{B}(J, K), 0 < j^2 u, k^2 v < X, \text{ and } m_1 j^2 u \equiv m_2 k^2 v \pmod{q} \}.$$

We also suppose

$$(6.5) \quad \max(|m_1|, |m_2|) \leq |m|.$$

In the following we estimate the general $\mathcal{N}[m_1, m_2](J, K)$ from which we can directly deduce an estimate for $\mathcal{N}[m](J, K)$ itself. The first bound we give is an auxiliary one and will be useful later on.

6.1. Auxiliary bound.

Lemma 6.1. *Let $X, q \geq 1$, q integer. Also let m, m_1, m_2 such that $0 < m_1, m_2 \leq |m|$. Let $\mathcal{N}[m_1, m_2](J, K)$ be as in (6.4), then for every $J, K \leq X$, we have*

$$(6.6) \quad \mathcal{N}[m_1, m_2] \ll_m \frac{X}{q} \left(X(JK)^{-1} + XJ^{-2}K \right) \mathcal{L},$$

where the implied constant depends at most on m .

Proof. We start by noticing that if we make $\ell q = m_1 j^2 u - m_2 k^2 v$, we have $|\ell| \leq \frac{2|m|X}{q}$. Hence we have the inequality

$$\mathcal{N}[m_1, m_2] \leq \sum_{|\ell| \leq \frac{2|m|X}{q}} \sum_{\substack{k \sim K \\ (k, q) = 1}} \sum_{u \leq XJ^{-2}} \sum_{\substack{j \sim J \\ m_1 j^2 u \equiv -\ell q \pmod{k^2} \\ (j, k) = 1}} 1.$$

Before we go any further we must do some considerations about greatest common divisor of the variables above. First we write $f = (j, k)$. Since $(k, q) = 1$, we must have $f^2 \mid \ell$. We then write

$$j_0 = \frac{j}{f}, k_0 = \frac{k}{f} \text{ and } \ell_0 = \frac{\ell}{f^2}.$$

The congruence above then becomes

$$m_1 j_0^2 u \equiv -\ell_0 q \pmod{k_0^2}.$$

Now, let $g = (k_0^2, m_1)$ as above we have $g \mid \ell_0$. We write

$$r = \frac{k_0^2}{g}, s = \frac{m_1}{g} \text{ and } t = \frac{\ell_0}{g}.$$

Finally, let $h = (r, t)$. From the considerations above, we must have $h \mid u$. We write

$$r' = \frac{r}{h}, t' = \frac{t}{h} \text{ and } u' = \frac{u}{h}.$$

So the congruence becomes

$$s u' j_0^2 \equiv -t' q \pmod{r'}$$

and since $(t' q, r') = 1$, it has at most $2.2^{\omega(r')} \leq 2d(k)$ solutions in $j_0 \pmod{t'}$. Therefore we have

$$\begin{aligned}
\mathcal{N}[m_1, m_2](J, K) &\leq \sum_{f \geq 1} \sum_{\ell_0 \leq \frac{X}{f^2 q}} \sum_{\substack{k_0 \sim K/f \\ (k_0, q)=1}} \sum_{u' \leq XJ^{-2}h^{-1}} \sum_{\substack{j_0 \sim J/f \\ su'j_0^2 \equiv -t'q \pmod{r'}}} 1 \\
&\leq 2 \sum_{f \geq 1} \sum_{\ell \leq \frac{X}{f^2 q}} \sum_{k_0 \sim K/f} XJ^{-2}h^{-1} \left\{ \frac{Jgh}{fk_0^2} + 1 \right\} d(k) \\
&\ll_m \sum_{f \geq 1} \sum_{\ell \leq \frac{X}{f^2 q}} \sum_{k_0 \sim K/f} XJ^{-2} \left\{ \frac{J}{fk_0^2} + 1 \right\} d(k) \\
&\ll_m \sum_{f \geq 1} \frac{X}{f^2 q} XJ^{-2} \left\{ \frac{J}{K^2} + 1 \right\} K\mathcal{L} \\
&\ll \frac{X}{q} XJ^{-2} \left\{ \frac{J}{K^2} + 1 \right\} K\mathcal{L}.
\end{aligned}$$

Hence the result. \square

Remark 6.1. We definitely have an inequality similar to (6.6) with the roles of J and K interchanged.

6.2. Square Sieve. We must now proceed to obtain a more precise bound. We start from the equality

$$\mathcal{N}[m_1, m_2](J, K) = \sum_{u \leq XJ^{-2}} \mathcal{N}_u[m_1, m_2](J, K),$$

where

$$\mathcal{N}_u[m_1, m_2](J, K) = \# \{ (j, k, v); j, k \in \mathcal{B}(J, K), 0 < k^2 v < X \text{ and } m_1 j^2 u \equiv m_2 k^2 v \pmod{q} \}.$$

In other words, we fix u and see how many times it contributes to $\mathcal{N}[m_1, m_2](J, K)$.

Again by dyadic decomposition, as we did in (6.2), we see that there is a certain U satisfying

$$(6.7) \quad U \leq XJ^{-2},$$

such that

$$(6.8) \quad \mathcal{N}[m_1, m_2](J, K) \ll \mathcal{N}[m_1, m_2](J, K, U)\mathcal{L},$$

where

$$\mathcal{N}[m_1, m_2](J, K, U) = \sum_{u \sim U} \mathcal{N}_u[m_1, m_2](J, K).$$

If we do an analysis of the possible values for v and $\ell = \frac{m_2 k^2 v - m_1 j^2 u}{q}$ as we count $\mathcal{N}_u[m_1, m_2](J, K)$, we obtain

$$\mathcal{N}_u[m_1, m_2](J, K) \leq \# \left\{ (j, k, \ell, v); j \sim J, k \sim K, m_1 j^2 u = m_2 k^2 v - \ell q, |\ell| \leq \frac{2|m|X}{q}, v \leq XK^{-2}, (j, k) = 1 \right\},$$

for every $u \sim U$.

We now appeal to the square sieve as in [6] that we state here for easier reference.

Theorem 6.2. ([6, Theorem 1]) *Let \mathcal{P} be a set of P odd primes and $(w(n))_{n \geq 1}$ a sequence of real numbers. Suppose that $w(n) = 0$ for $n = 0$ or $n \geq e^P$. Then*

$$\sum_{n \geq 1} w(n^2) \ll P^{-1} \sum_{n \geq 1} w(n) + P^{-2} \sum_{\substack{p_1 \neq p_2 \\ p_1, p_2 \in \mathcal{P}}} \left| \sum_n w(n) \left(\frac{n}{p_1 p_2} \right) \right|,$$

where $\left(\frac{n}{p_1 p_2} \right)$ is the Jacobi symbol and the implied constant is absolute.

We apply the square-sieve to the multi-set of integers

$$\mathcal{A}_u = \left\{ \frac{m_1(m_2 k^2 v - \ell q)}{u}; u \mid m_2 k^2 v - \ell q, k \sim K, \ell \leq \frac{2|m|X}{q}, v \leq X K^{-2} \right\}.$$

For n integer, let $w_u(n)$ denote the number of times where n appears in \mathcal{A}_u . We have

$$\mathcal{N}_u[m_1, m_2](J, K) \leq \sum_{j \geq 1} w_u((m_1 j)^2) \leq \sum_{n \geq 1} w_u(n^2).$$

For the set of primes, we take

$$\mathcal{P} = \left\{ p \text{ prime} ; p \nmid 2m_1 m_2 u, \hat{P} < p \leq 2\hat{P} \right\},$$

where \hat{P} will be chosen later depending on J, K, X, q , subject to the condition

$$(6.9) \quad (\log X)^2 \leq \hat{P} \leq X.$$

We have $P = \#\mathcal{P} \sim \hat{P}(\log \hat{P})^{-1}$ as $\hat{P} \rightarrow \infty$ and thus

$$w_u(n) = 0 \text{ for } n \geq e^P,$$

because, for X sufficiently large,

$$e^P \geq e^{\hat{P}^{1/2}} \geq e^{\log X} = X,$$

and $w_u(n) = 0$ for $n > X$.

Theorem 6.2 and the definition of \mathcal{P} then give the inequality

$$(6.10) \quad \mathcal{N}_u[m_1, m_2](J, K) \ll \hat{P}^{-1}(\log X) \sum_{n \geq 1} w_u(n) + P^{-2} \sum_{\substack{p_1, p_2 \\ p_1, p_2 \in \mathcal{P} \\ p_1 \neq p_2}} \left| \sum_{k, \ell, v} \left(\frac{m_1 u(m_2 k^2 v - \ell q)}{p_1 p_2} \right) \right|,$$

where the conditions in the last sum are

$$(6.11) \quad k \sim K, \ell \leq \frac{X}{q}, v \leq X K^{-2}, u \mid m_2 k^2 v - \ell q.$$

To simplify, we write

$$T_1(u) = \hat{P}^{-1}(\log X) \sum_{n \geq 1} w_u(n)$$

$$T_2(u) = P^{-2} \sum_{\substack{p_1, p_2 \\ p_1, p_2 \in \mathcal{P}}} \left| \sum_{k, \ell, v} \left(\frac{u(m_2 k^2 v - \ell q)}{p_1 p_2} \right) \right|.$$

So that (6.10) implies the inequality

$$(6.12) \quad \mathcal{N}[m_1, m_2](J, K, U) \ll \sum_{u \sim U} T_1(u) + \sum_{u \sim U} T_2(u).$$

6.2.1. *Study of $T_1(u)$.* Because of the definition of $w_u(n)$, we have

$$\begin{aligned} \sum_{u \sim U} T_1(u) &\ll \widehat{P}^{-1}(\log X) \sum_{k, \ell, v} \sum_{u | m_2 k^2 v - \ell q} 1 \\ &\ll \widehat{P}^{-1}(\log X) \sum_{k, \ell, v} d(m_2 k^2 v - \ell q), \end{aligned}$$

where the conditions on the sum are

$$(6.13) \quad k \sim K, \ell \leq \frac{X}{q}, v \leq X K^{-2}, m_2 k^2 v - \ell q \geq 1.$$

For the sum above, we appeal to Lemma 9.2 below, giving the inequality

$$(6.14) \quad \sum_{u \sim U} T_1(u) \ll_{\eta} \frac{X}{q} (X^{1/2+\eta} + X(K\widehat{P})^{-1} \mathcal{L}^3),$$

for every $\eta > 0$, where the implied constant depends only on η .

Note that for the first term of (6.14), we used the inequality $\widehat{P}^{-1} \log X \leq 1$ (see (6.9)).

6.2.2. *Study of $T_2(u)$.* We have

$$\begin{aligned} \sum_{u \sim U} T_2(u) &= P^{-2} \sum_{p_1 \neq p_2} \sum_{u \sim U} \left| \sum_{k, \ell, v} \left(\frac{m_1 u (m_2 k^2 v - \ell q)}{p_1 p_2} \right) \right| \\ (6.15) \quad &\leq \sum_{u \sim U} \max_{\substack{p_1 \nmid 2m_1 m_2 u \\ \widehat{P} < p_1 < p_2 \leq 2\widehat{P}}} \left| \sum_{k, \ell, v} \left(\frac{(m_2 k^2 v - \ell q)}{p_1 p_2} \right) \right|. \end{aligned}$$

We pick $(p_1, p_2) = (p_1(u), p_2(u))$ for which the maximum is attained, and we proceed to estimate the sum

$$(6.16) \quad S_u := \sum_{k, \ell, v} \left(\frac{m_2 k^2 v - \ell q}{p_1 p_2} \right),$$

where the conditions on k, ℓ, v are as in (6.11). And we recall $p_1 \neq p_2$.

Thus, (6.15) becomes

$$(6.17) \quad \sum_{u \sim U} T_2(u) \ll \sum_{u \sim U} |S_u|$$

The following calculations follow the same lines of the main calculations in the proof of [6, Theorem 2]. The main difference is the additional sum over ℓ which induces big cancellations. We write

$$\begin{aligned}
S_u &= \sum_{\substack{\alpha, \beta, \gamma=1 \\ u|m_2 \alpha^2 \beta - q\gamma}}^{up_1 p_2} \left(\frac{\alpha^2 \beta - q\gamma}{p_1 p_2} \right) \left| \sum_{\substack{k \sim K \\ k \equiv \alpha \pmod{up_1 p_2}}} 1 \right| \left| \sum_{\substack{v \leq XK^{-2} \\ v \equiv \beta \pmod{up_1 p_2}}} 1 \right| \left| \sum_{\substack{\ell \leq X/q \\ \ell \equiv \gamma \pmod{up_1 p_2}}} 1 \right| \\
&= \sum_{\substack{\alpha, \beta, \gamma \\ u|m_2 \alpha^2 \beta - q\gamma}} \left(\frac{\alpha^2 \beta - q\gamma}{p_1 p_2} \right) \left\{ \frac{1}{up_1 p_2} \sum_{\lambda=1}^{up_1 p_2} \sum_{k \sim K} e \left(\frac{\lambda(\alpha - k)}{up_1 p_2} \right) \right\} \left\{ \frac{1}{up_1 p_2} \sum_{\mu=1}^{up_1 p_2} \sum_{v \leq XK^{-2}} e \left(\frac{\mu(\beta - v)}{up_1 p_2} \right) \right\} \\
&\quad \cdot \left\{ \frac{1}{up_1 p_2} \sum_{\nu=1}^{up_1 p_2} \sum_{\ell \leq \frac{X}{q}} e \left(\frac{\nu(\gamma - \ell)}{up_1 p_2} \right) \right\} \\
(6.18) \quad &= (up_1 p_2)^{-3} \sum_{\lambda, \mu, \nu=1}^{up_1 p_2} S(u, p_1 p_2, q, m_2; \lambda, \mu, \nu) \Theta_\lambda \Phi_\mu \Psi_\nu,
\end{aligned}$$

where

$$(6.19) \quad \begin{cases} S(u, p_1 p_2, q, m_2; \lambda, \mu, \nu) = \sum_{\substack{\alpha, \beta, \gamma=1 \\ u|m_2 \alpha^2 \beta - q\gamma}}^{up_1 p_2} \left(\frac{\alpha^2 \beta - q\gamma}{p_1 p_2} \right) e \left(\frac{\lambda \alpha + \mu \beta + \nu \gamma}{up_1 p_2} \right), \\ \Theta_\lambda = \sum_{K < k \leq 2K} e \left(\frac{-\lambda k}{up_1 p_2} \right) \ll \min \left(K, \left\| \frac{\lambda}{up_1 p_2} \right\|^{-1} \right), \\ \Phi_\mu = \sum_{v \leq XK^{-2}} e \left(\frac{-\mu v}{up_1 p_2} \right) \ll \min \left(XK^{-2}, \left\| \frac{\mu}{up_1 p_2} \right\|^{-1} \right), \\ \Psi_\nu = \sum_{\ell \leq \frac{X}{q}} e \left(\frac{-\nu \ell}{up_1 p_2} \right) \ll \min \left(\frac{X}{q}, \left\| \frac{\nu}{up_1 p_2} \right\|^{-1} \right), \end{cases}$$

where $\|x\|$ denotes the distance from x to the nearest integer.

The chinese remainder theorem allows us to write

$$(6.20) \quad S(u, p_1 p_2; \lambda, \mu, \nu) = S_1(p_1; b, c, d) S_1(p_2; b, c, d) \prod_{r^f \parallel u} S_2(r^f; b, c, d),$$

where $u = \prod r^f$ is the prime factorization of u , and b, c, d are some integers such that

$$\begin{cases} (b, up_1 p_2) = (\lambda, up_1 p_2), \\ (c, up_1 p_2) = (\mu, up_1 p_2), \\ (d, up_1 p_2) = (\nu, up_1 p_2) \end{cases}$$

and

$$\begin{cases} S_1(p, q, m_2; b, c, d) = \sum_{\alpha, \beta, \gamma=1}^p \left(\frac{m_2 \alpha^2 \beta - q\gamma}{p} \right) e \left(\frac{b\alpha + c\beta + d\gamma}{p} \right) \\ S_2(r^f, q, m_2; b, c, d) = \sum_{\substack{\alpha, \beta, \gamma=1 \\ r^f | m_2 \alpha^2 \beta - q\gamma}}^{r^f} e \left(\frac{b\alpha + c\beta + d\gamma}{r^f} \right). \end{cases}$$

For these sums we have the following bounds whose proofs are elementary and we postpone until section 9 (see Lemma 9.3).

$$\begin{cases} S_1(p, q, m_2; b, c, 0) = 0, \\ S_1(p, q, m_2; b, c, d) \ll p^{3/2}, \\ |S_2(r, q, m_2; b, c, d)| \leq 2r(r, b, c, dm_2) \\ |S_2(r^f, q, m_2; b, c, d)| \leq 2r^{\frac{3f}{2}}(r^f, b, c, dm_2), \text{ if } r \text{ is odd, } f \geq 2, \\ |S_2(2^f, q, m_2; b, c, d)| \leq 4.2^{\frac{3f}{2}}(2^f, b, c, dm_2), f \geq 2. \end{cases}$$

If we multiply these upper bounds in (6.20), we have, whenever $d \not\equiv 0 \pmod{up_1p_2}$,

$$(6.21) \quad \begin{aligned} S(u, p_1p_2, m_2; b, c, d) &\ll d(u) \hat{P}^3 U^{3/2} (u^\dagger)^{-1/2} (u, b, c, dm_2) \\ &\ll_m d(u) \hat{P}^3 U^{3/2} (u^\dagger)^{-1/2} (u, b, c, d), \end{aligned}$$

where

$$(6.22) \quad u^\dagger = \prod_{\substack{p|u \\ p^2 \nmid u}} p.$$

and

$$S(u, p_1p_2, m_2; b, c, 0) = 0.$$

Hence, by (6.18), we deduce the inequality

$$(6.23) \quad S_u \ll_m d(u) \hat{P}^{-3} U^{-3/2} (u^\dagger)^{-1/2} \sum_{\substack{\lambda, \mu, \nu \pmod{up_1p_2} \\ \nu \not\equiv 0 \pmod{up_1p_2}}} |\Theta_\lambda \Phi_\mu \Psi_\nu|(u, \lambda, \mu, \nu).$$

Now, we separate the above triple sum in eight parts accordingly to whether λ, μ and ν are zero or not. We also use the bounds coming from (6.19). We use the first term inside the min-symbol in the zero case and the second one for the others. We then have

$$(6.24) \quad \begin{aligned} \sum_{\lambda, \mu, \nu \pmod{up_1p_2}} |\Theta_\lambda \Phi_\mu \Psi_\nu|(u, \lambda, \mu, \nu) &\ll K.XK^{-2}.U\hat{P}^2 \sum_{1 \leq \nu \leq \frac{up_1p_2}{2}} \nu^{-1}(u, \nu) + K(U\hat{P}^2)^2 \sum_{1 \leq \mu, \nu \leq \frac{up_1p_2}{2}} \mu^{-1}\nu^{-1}(u, \mu, \nu) \\ &+ XK^{-2}(U\hat{P}^2)^2 \sum_{1 \leq \lambda, \nu \leq \frac{up_1p_2}{2}} \lambda^{-1}\nu^{-1}(u, \lambda, \nu) + (U\hat{P}^2)^3 \sum_{1 \leq \lambda, \mu, \nu \leq \frac{up_1p_2}{2}} \lambda^{-1}\mu^{-1}\nu^{-1}(u, \lambda, \mu, \nu). \end{aligned}$$

For the sums involving greatest common divisors, we have the following Lemma

Lemma 6.3. *Let $Z \geq 1$ and u an integer such that $u \geq 1$, then we have the following inequalities*

$$(6.25) \quad \sum_{1 \leq \lambda \leq Z} \lambda^{-1}(u, \lambda) \leq d(u) (\log(2Z)),$$

$$(6.26) \quad \sum_{1 \leq \lambda, \mu \leq Z} \lambda^{-1}\mu^{-1}(u, \lambda, \mu) \leq d(u) (\log(2Z))^2,$$

$$(6.27) \quad \sum_{1 \leq \lambda, \mu, \nu \leq Z} \lambda^{-1}\mu^{-1}\nu^{-1}(u, \lambda, \mu, \nu) \ll (\log 2Z)^3.$$

Proof. We prove (6.27), the other ones are analogous. We have the following

$$\begin{aligned}
\sum_{1 \leq \lambda, \mu, \nu \leq Z} \lambda^{-1} \mu^{-1} \nu^{-1} (u, \lambda, \mu, \nu) &\leq \sum_{d|u} d \sum_{\substack{1 \leq \lambda, \mu, \nu \leq Z \\ d|\lambda, \mu, \nu}} \lambda^{-1} \mu^{-1} \nu^{-1} \\
&\leq \sum_{d|u} d \sum_{1 \leq \lambda', \mu', \nu' \leq \frac{Z}{d}} d^{-3} \lambda'^{-1} \mu'^{-1} \nu'^{-1} \\
&= \sum_{d|u} d^{-2} \left(\sum_{1 \leq \lambda' \leq \frac{Z}{d}} \lambda'^{-1} \right) \\
&\ll (\log(2Z))^3
\end{aligned}$$

□

If we insert the inequalities (6.25), (6.26) and (6.27) with $Z = \frac{up_1 p_2}{2}$ in (6.24) and notice that $\log(up_1 p_2) \leq 3\mathcal{L}$, we deduce

$$(6.28) \quad S_u \ll_m d(u)^2 (u^\dagger)^{-1/2} \mathcal{L}^3 \left\{ K^{-1} \hat{P}^{-1} U^{-1/2} X + K \hat{P} U^{1/2} + K^{-2} \hat{P} U^{1/2} X + \hat{P}^3 U^{3/2} \right\}.$$

in order to compute the contribution of S to $\mathcal{N}[m_1, m_2](J, K, U)$ we must sum over u . To do so, we present the following

Lemma 6.4. *Let $U \geq 1$ be a real number. For each u integer, such that $u \geq 1$, let u^\dagger be its squarefree part as in (6.22). Then we have*

$$(6.29) \quad \sum_{u \sim U} d(u)^2 (u^\dagger)^{-1/2} \ll U^{1/2} (\log(2U))^3,$$

where the implied constant is absolute.

Proof. It is sufficient to prove that

$$\sum_{u \leq U} d(u)^2 (u^\dagger)^{-1/2} u^{1/2} \ll U (\log U)^3.$$

Considering the Dirichlet series

$$F(s) = \sum_{n \geq 1} d(n)^2 (n^\dagger)^{-1/2} n^{1/2} n^{-s}$$

and expressing it in an Euler product, we see that $F(s) \zeta^{-4}(s)$ is holomorphic in the half plane

$$\operatorname{Re}(s) > \sigma, \text{ for some } \sigma < 1.$$

Hence, $F(s)$ has a pole of order at most 4 at $s = 1$. this gives the result. □

Gathering (6.17), (6.28), and summing over u by Lemma 6.4, we obtain

$$\sum_{u \sim U} T_2(u) \ll_m \left\{ K^{-1} \hat{P}^{-1} X + K \hat{P} U + K^{-2} \hat{P} U X + \hat{P}^3 U^2 \right\} \mathcal{L}^6$$

Appealing to (6.7), we have the inequality

$$(6.30) \quad \sum_{u \sim U} T_2(u) \ll_m \left\{ K^{-1} \hat{P}^{-1} X + J^{-2} K \hat{P} X + J^{-2} K^{-2} \hat{P} X^2 + J^{-4} \hat{P}^3 X^2 \right\} \mathcal{L}^6.$$

From (6.12), (6.14) and (6.30), we deduce an inequality for $\mathcal{N}[m_1, m_2](J, K, U)$

$$\mathcal{N}[m_1, m_2](J, K, U) \ll_{\eta, m} \left\{ X^{3/2+\eta} q^1 + K^{-1} \hat{P}^{-1} X^2 q^{-1} + K^{-1} \hat{P}^{-1} X \right. \\ \left. + J^{-2} K \hat{P} X + J^{-2} K^{-2} \hat{P} X^2 + J^{-4} \hat{P}^3 X^2 \right\} \mathcal{L}^6.$$

Since $\frac{X}{q} \geq 1$, the third term above can be seen to be

$$\leq K^{-1} \hat{P}^{-1} X^2 q^{-1}.$$

And therefore,

$$(6.31) \quad \mathcal{N}[m_1, m_2](J, K, U) \ll_{\eta, m} \left\{ X^{3/2+\eta} q^1 + K^{-1} \hat{P}^{-1} X^2 q^{-1} + J^{-2} K \hat{P} X \right. \\ \left. + J^{-2} K^{-2} \hat{P} X^2 + J^{-4} \hat{P}^3 X^2 \right\} \mathcal{L}^6.$$

Remark 6.2. We recall the importance of working with the general $\mathcal{N}[m_1, m_2](J, K)$ instead of $\mathcal{N}[m](J, K)$. The reason is that because of the equalities

$$\mathcal{N}[m](J, K) = \mathcal{N}[m, 1](J, K) = \mathcal{N}[1, m](K, J),$$

the upper bound (6.31) still holds if we replace the roles of J and K . in other words, there is no loss of generality in supposing $J \geq K$.

Now, the upper bound (6.31), together with (6.3) and (6.8) gives us

$$(6.32) \quad S_{>y}[m](X, q) \ll_{\eta, m} \left\{ X^{3/2+\eta} q^1 + K^{-1} \hat{P}^{-1} X^2 q^{-1} + J^{-2} K \hat{P} X \right. \\ \left. + J^{-2} K^{-2} \hat{P} X^2 + J^{-4} \hat{P}^3 X^2 \right\} \mathcal{L}^9.$$

At this point we make the choice

$$(6.33) \quad \hat{P} = JK^{-1/4} q^{-1/4} + \mathcal{L}^2,$$

which makes the second and the last terms of (6.32) similar. Hence we have

$$S_{>y}[m](X, q) \ll_{\eta} \left\{ X^{3/2+\eta} q^{-1} + J^{-1} K^{-3/4} X^2 q^{-3/4} + J^{-1} K^{3/4} X q^{-1/4} \right. \\ \left. + J^{-1} K^{-9/4} X^2 q^{-1/4} + J^{-2} K X + J^{-2} K^{-2} X^2 + J^{-4} X^2 \right\} \mathcal{L}^{15}.$$

Since $J \geq K$, $JK \geq y$, we have

$$(6.34) \quad S_{>y}[m](X, q) \ll_{\eta} \left\{ X^{3/2+\eta} q^{-1} + X^2 q^{-3/4} y^{-7/8} + X q^{-1/4} y^{-1/8} \right. \\ \left. + J^{-1} K^{-9/4} X^2 q^{-1/4} + X y^{-1/2} + X^2 y^{-2} \right\} \mathcal{L}^{15}.$$

7. PROOF OF THEOREM 1.4

We assume

$$(7.1) \quad y \geq X^{\delta}, \text{ for some } \delta > 1/2,$$

and we choose η sufficiently small so that we have

$$\eta < \delta - \frac{1}{2}.$$

We go back to (3.6). Since we have studied $S_{\leq y}[m](X, q)$ (see (3.16)) and $S_{<y}[m](X, q)$ (see (6.34)), we deduce

$$(7.2) \quad S[m](X, q) = \mathcal{A}[m](X, q) + R[m](X, q)$$

where $R[m](x, q)$ satisfies

$$R[m](X, q) \ll_m \left\{ d(q)Xyq^{-1} + X^2y^{-7/8}q^{-3/4} + Xy^{-1/8}q^{-1/4} \right. \\ \left. + J^{-1}K^{-9/4}X^2q^{-1/4} + Xy^{-1/2} + X^2y^{-2} \right\} \mathcal{L}^{15},$$

and the missing term can be seen to be $\ll d(q)Xyq^{-1}\mathcal{L}^{15}$ thanks to (7.1). We now make the choice

$$(7.3) \quad y = X^{8/15}q^{2/15}$$

to make the two first terms similar.

Notice that this choice of y implies that (7.1) is satisfied. Hence we have

$$(7.4) \quad R[m](X, q) \ll_m \left\{ d(q)X^{23/15}q^{-13/15} + X^{14/15}q^{-4/15} + J^{-1}K^{-9/4}X^2q^{-1/4} \right. \\ \left. + X^{11/15}q^{-1/15} + X^{14/15}q^{-4/15} \right\} \mathcal{L}^{15} \\ \ll \left\{ d(q)X^{23/15}q^{-13/15} + \mathcal{E}_1 \right\} \mathcal{L}^{15},$$

where

$$\mathcal{E}_1 = X^2J^{-1}K^{-9/4}q^{-1/4}$$

and the missing terms are $\leq d(q)X^{23/15}q^{-13/15}$, since $q \leq X$.

At this point, we recall the auxiliary bound from lemma 6.1. The inequality (6.6), together with (3.6), (3.16), (6.3) and, of course, the choice $y = X^{8/15}q^{2/15}$ give us

$$(7.5) \quad R[m](X, q) \ll_m \left\{ d(q)Xyq^{-1} + X^2J^{-1}K^{-1}q^{-1} + X^2J^{-2}Kq^{-1} \right\} \mathcal{L}^3 \\ \ll \left\{ d(q)Xyq^{-1} + X^2y^{-1}q^{-1} + \mathcal{E}_2 \right\} \mathcal{L}^3 \\ \ll \left\{ d(q)X^{23/15}q^{-13/15} + \mathcal{E}_2 \right\} \mathcal{L}^3,$$

where, again the missing term can be seen to be $\ll d(q)Xyq^{-1}\mathcal{L}^3$ thanks to (7.1) and

$$\mathcal{E}_2 = X^2J^{-2}Kq^{-1}.$$

The importance of the auxiliary bound (6.6) is that since $J \geq K$, we cannot give a good estimate for the term \mathcal{E}_1 using that $JK \geq y$. So we look for a mixed term that lies between \mathcal{E}_1 and \mathcal{E}_2 for which we have a good bound (much better than the one for \mathcal{E}_2 , for example).

Notice that

$$(7.6) \quad \min(\mathcal{E}_1, \mathcal{E}_2) \leq \mathcal{E}_1^{12/17} \mathcal{E}_2^{5/17} \\ = X^2(JK)^{-22/17}q^{-8/17} \\ \ll X^2y^{-22/17}q^{-8/17} \\ \ll X^{334/255}q^{-164/255}.$$

Now, (7.4), (7.5), (7.6) imply

$$(7.7) \quad \begin{aligned} R[m](X, q) &\ll_m \left\{ d(q)X^{23/15}q^{-13/15} + X^{334/255}q^{-164/255} \right\} \mathcal{L}^{15} \\ &\ll d(q)X^{23/15}q^{-13/15} \mathcal{L}^{15}, \end{aligned}$$

where the missing term is $\leq d(q)X^{23/15}q^{-13/15}$ since $q \leq X$.

The formula (1.17) now follows from (5.1) and the bound (7.7) for $R[m](X, q)$ above. The proof of Theorem 1.4 is complete.

8. PROOF OF THEOREM 1.1

In this section we prove (1.7) as a corollary of theorem 1.4.

If we take a look at (1.14), we see that the only term we still need to estimate is

$$\sum_{\substack{n \leq X \\ (n, q)=1}} \mu^2(n).$$

The following is a very simple case of [1, Lemma 3.1.].

Lemma 8.1. *Let $\epsilon > 0$ be given. We then have the equality*

$$(8.1) \quad \sum_{\substack{n \leq X \\ (n, q)=1}} \mu^2(n) = \frac{\varphi(q)}{q} \frac{6}{\pi^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{-1} X + O\left(X^{1/2+\epsilon}\right),$$

uniformly for $1 \leq q \leq X$, where the O -constant depends on ϵ alone.

We now prove Theorem 1.1 by substituting equation (8.1) in (1.14). We obtain the equality

$$\mathcal{M}_2[m](X, q) = 2S[m](X, q) - \varphi(q) \left(C(q) \frac{X}{q}\right)^2 + O(X^{3/2+\epsilon}q^{-1}).$$

Now, we use (1.17) for the term $S[m](X, q)$ and the main term disappears. So we deduce

$$\begin{aligned} \mathcal{M}_2[m](X, q) &= \frac{C}{2} \Gamma_{\text{an}}(m) \Gamma_{\text{ar}}(m) \prod_{p|q} \left(1 + 2p^{-1}\right)^{-1} X^{1/2} q^{1/2} \\ &\quad + O_{\epsilon, m}(d(q)X^{1/3}q^{2/3} + d(q)X^{23/15}q^{-13/15} \mathcal{L}^{15} + X^{3/2+\epsilon}q^{-1}). \end{aligned}$$

By comparing the error terms, if we choose ϵ small enough ($\epsilon < 1/30$ suffices), then the last error term is smaller than the second one. This concludes the proof of (1.7).

9. APPENDIX

9.1. Sums involving the divisor function. In this appendix we analyze a certain sum involving the divisor function which we used in the corresponding proof. For that purpose, we use the following classical Lemma that can be found in [11], for instance.

Lemma 9.1. *Let $a, q \in \mathbb{Z}$, $q > 0$ such that $(a, q) = 1$, let $X > 0$ and $\eta > 0$ be given. Then*

$$\sum_{\substack{n \leq X \\ n \equiv a \pmod{q}}} d(n) \ll_{\eta} \frac{\varphi(q)}{q^2} X (\log 2X),$$

uniformly for $q \leq X^{1-\eta}$. The implied constant in \ll_{η} depending on η alone.

We need to estimate the sum

$$\sum_{k,v,\ell} d(k^2v - \ell q),$$

where the conditions on the sum are as in (6.13).

For that we have the more general result

Lemma 9.2. *Let $q \geq 1$ be an integer and let $\eta > 0$ be given. Then uniformly for $K, S, X \geq 1$ real numbers such that $S \leq XK^{-2}$, we have*

$$\sum_{k \sim K} \sum_{\ell \leq \frac{X}{q}} \sum_{\substack{v \leq S \\ k^2v - \ell q \geq 1}} d(k^2v - \ell q) \ll_{\eta} \frac{X}{q} \left(X^{1/2+\eta} + XK^{-1}\mathcal{L}^3 \right),$$

where the implied constant depends only on η .

Proof. Let $f = (k^2, \ell q)$. We write $k^2 = fg$, $\ell q = fh$.

Let $\eta > 0$. We divide in two cases

i) If $K \leq \frac{1}{2}X^{1/2-\eta/2}$, then $k^2 \leq X^{1-\eta}$ and also $\frac{k^2}{f} \leq \frac{X^{1-\eta}}{f} \leq \left(\frac{4X}{f} \right)^{1-\eta}$. So by Lemma 9.2 above,

$$\begin{aligned} \sum_{\substack{v \leq S \\ k^2v - \ell q \geq 1}} d(k^2v - \ell q) &\leq \sum_{v \leq S} d(f)d(gv - h) \\ &\ll d(f) \sum_{\substack{n \leq \frac{k^2S}{f} \\ n \equiv -h \pmod{g}}} d(n) \\ &\ll d(f) \sum_{\substack{n \leq \frac{4X}{f} \\ n \equiv -h \pmod{g}}} d(n) \\ &\ll_{\eta} d(f)\varphi(g) \frac{X}{fg^2} \mathcal{L}. \end{aligned}$$

So, since $\varphi(g) \leq g$, $f \mid k^2$ and $fg = k^2$, we obtain

$$(9.1) \quad \sum_{\substack{v \leq S \\ k^2v - \ell q \geq 1}} d(k^2v - \ell q) \ll_{\eta} d(k^2)XK^{-2}\mathcal{L}.$$

ii) If $K > \frac{1}{2}X^{1/2-\eta/2}$, we have that $X/K \leq 2X^{1/2+\eta/2}$.

Then, by the classical bound $d(n) \ll_{\eta} n^{\eta}$, we obtain

$$\begin{aligned} \sum_{\substack{v \leq S \\ k^2v - \ell q \geq 1}} d(k^2v - \ell q) &\ll_{\eta} S(k^2v - \ell q)^{\eta/2} \\ &\ll \frac{X}{K^2} X^{\eta/2} \\ (9.2) \quad &\ll X^{1/2+\eta} K^{-1}. \end{aligned}$$

Now, if we put the two cases together, (9.1) and (9.2) lead to

$$\sum_{\substack{v \leq S \\ k^2v - \ell q \geq 1}} d(k^2v - \ell q) \ll_{\eta} X^{1/2+\eta} K^{-1} + d(k^2)XK^{-2}\mathcal{L}.$$

After summation over k and ℓ , we finally obtain

$$\begin{aligned} \sum_{k,\ell,v} d(k^2v - \ell q) &\ll_{\eta} \sum_{k \sim K} \sum_{\ell \leq \frac{X}{q}} \left(X^{1/2+\eta} K^{-1} + d(k^2) X K^{-2} \mathcal{L} \right) \\ &\ll_{\eta} \frac{X}{q} \left(X^{1/2+\eta} + X K^{-1} \mathcal{L}^3 \right). \end{aligned}$$

□

9.2. Exponential Sums. Here we give the proof for the bounds on exponential sums. We have the following

Lemma 9.3. *Let p, q, m_2, b, c, d be integers, $m_2 \neq 0$, p a prime number and $p \nmid m_2 q$. We define*

$$S_1(p, q, m_2; b, c, d) = \sum_{\alpha, \beta, \gamma=1}^p \left(\frac{m_2 \alpha^2 \beta - q \gamma}{p} \right) e \left(\frac{b \alpha + c \beta + d \gamma}{p} \right).$$

Let r be a prime number such that $r \nmid q$ and $f > 0$ an integer,

$$S_2(r^f, q, m_2; b, c, d) = \sum_{\substack{\alpha, \beta, \gamma=1 \\ r^f \mid m_2 \alpha^2 \beta - q \gamma}}^{r^f} e \left(\frac{b \alpha + c \beta + d \gamma}{r^f} \right).$$

Then, we have

$$\begin{aligned} (9.3) \quad & S_1(p, q, m_2; b, c, 0) = 0, \\ (9.4) \quad & S_1(p, q, m_2; b, c, d) \ll p^{3/2}, \\ (9.5) \quad & |S_2(r, q, m_2; b, c, d)| \leq 2r(r, b, c, dm_2), \\ (9.6) \quad & |S_2(r^f, q, m_2; b, c, d)| \leq 2r^{\frac{3f}{2}}(r^f, b, c, dm_2)^{\frac{1}{2}} \text{ if } r \text{ is odd, } f \geq 2, \\ (9.7) \quad & |S_2(2^f, q, m_2; b, c, d)| \leq 4.2^{\frac{3f}{2}}(2^f, b, c, dm_2)^{\frac{1}{2}}, \quad f \geq 2. \end{aligned}$$

Proof. We start by the study of S_2 .

We have

$$\begin{aligned} S_2(r, q, m_2; b, c, d) &= \sum_{\alpha, \beta=1}^r e \left(\frac{b \alpha + c \beta + d \bar{q} m_2 \alpha^2 \beta}{r} \right) \\ &= \sum_{\alpha=1}^r e \left(\frac{b \alpha}{r} \right) \sum_{\beta=1}^r e \left(\frac{(c + d m_2 \bar{q} \alpha^2) \beta}{r} \right) \\ (9.8) \quad &= \sum_{\alpha=1}^r e \left(\frac{b \alpha}{r} \right) \delta(c + d m_2 \bar{q} \alpha^2, r), \end{aligned}$$

where

$$\delta(x, n) = \begin{cases} n & \text{if } n \mid x, \\ 0 & \text{otherwise.} \end{cases}$$

If $r \mid d m_2$, equation (9.8) becomes

$$S_2(r, q, m_2; b, c, d) = \sum_{\alpha=1}^r e \left(\frac{b \alpha}{r} \right) \delta(c, r) = \delta(b, r) \delta(c, r).$$

So, $S_2(r, q; b, c, d) = 0$ unless r divides both b and c , in which case,

$$S_2(r, q, m_2; b, c, d) = r^2 = r(r, b, c, dm_2).$$

That proves (9.5) when $r \mid dm_2$.

We now assume $r \nmid dm_2$. We analyze when the symbol $\delta(c + dm_2 \bar{q} \alpha^2)$ is non-zero. That means we consider the equation

$$dm_2 \alpha^2 \equiv -cq \pmod{r}.$$

This equation has at most two solutions for $1 \leq \alpha \leq r$. Thus, again by equation (9.8),

$$|S_2(r, q, m_2; b, c, d)| = \left| \sum_{\alpha=1}^r e\left(\frac{b\alpha}{r}\right) \delta(c + dm_2 \bar{q} \alpha^2, r) \right| \leq 2r,$$

which completes the proof of (9.5) in Lemma 9.3.

We proceed to prove the inequalities (9.6) and (9.7). Analogously to the previous case, we have

$$S_2(r^f, q, m_2; b, c, d) = \sum_{\alpha=1}^{r^f} e\left(\frac{b\alpha}{r^f}\right) \delta(c + dm_2 \bar{q} \alpha^2, r^f).$$

We write $(c, r^f) = r^s$, $(dm_2, r^f) = r^t$. So we have $0 \leq s, t \leq f$

If $s < t$, for any α , the biggest power of r that divides $c + dm_2 \bar{q} \alpha^2$ is always r^s . So the δ symbol is always zero. Hence the sum is itself always zero.

So, we suppose $s \geq t$.

In this case, we write $c = r^s \tilde{c}$, $dm_2 = r^t \tilde{d}$. Then, the condition $r^f \mid c + dm_2 \bar{q} \alpha^2$ is equivalent to

$$(9.9) \quad r^{f-t} \mid r^{s-t} \tilde{c} + \tilde{d} \bar{q} \alpha^2.$$

Notice that for any α for which (9.9) is true, we must have

$$r^u \mid \alpha,$$

where $u = \lceil \frac{s-t}{2} \rceil$. We write $\alpha = r^u \tilde{\alpha}$, with $1 \leq \tilde{\alpha} \leq r^{f-u}$. Condition (9.9) now translates to

$$(9.10) \quad r^{f-s} \mid \tilde{c} + r^{2u-s+t} \tilde{d} \bar{q} \tilde{\alpha}^2,$$

which has at most 2 solutions for $\tilde{\alpha} \pmod{r^{f-s}}$, if r is odd.

Remark 9.1. *If $r = 2$ the quadratic equation above can have up to 4 solutions $\pmod{r^{f-s}}$, which is in fact the only difference between the cases r odd and $r = 2$.*

In the following, we write down the proof of (9.6). The proof of (9.7) is exactly the same but taking into account Remark 9.1. We now have

$$\begin{aligned}
S_2(r^f, q, m_2; b, c, d) &= r^f \sum_{\tilde{\alpha}=1}^{r^{f-u}}{}' e\left(\frac{br^u \tilde{\alpha}}{r^f}\right) \\
&= r^f \sum_{\tilde{\alpha}=1}^{r^{f-u}}{}' e\left(\frac{b\tilde{\alpha}}{r^{f-u}}\right) \\
&= r^f \sum_{\tilde{\alpha}=1}^{r^{f-s}}{}' e\left(\frac{b\tilde{\alpha}}{r^{f-u}}\right) \sum_{h=0}^{r^{s-u}} e\left(\frac{br^{f-s}h}{r^{f-u}}\right) \\
(9.11) \quad &= r^f \delta(b, r^{s-u}) \sum_{\tilde{\alpha}=1}^{r^{f-s}}{}' e\left(\frac{b\tilde{\alpha}}{r^{f-u}}\right),
\end{aligned}$$

where the $'$ in the sum means that we only sum over the $\tilde{\alpha}$ satisfying (9.10). From (9.11), we see that $S_2(r^f, q, m_2; b, c, d)$ is zero unless

$$(9.12) \quad r^{s-u} \mid b.$$

So, we assume (9.12). Then, (9.11) gives the inequality

$$(9.13) \quad |S_2(r^f, q, m_2; b, c, d)| \leq 2r^{f+s-u}.$$

And since $u \geq \frac{s-t}{2}$,

$$f + s - u \leq f + \frac{s+t}{2} \leq \frac{3f}{2} + \frac{t}{2}.$$

As a consequence, (9.13) becomes

$$(9.14) \quad |S_2(r^f, q, m_2; b, c, d)| \leq 2r^{\frac{3f}{2}} (r^t)^{\frac{1}{2}}.$$

We want to prove that

$$(9.15) \quad (r^f, b, c, dm_2) = r^t.$$

Since $t \leq s$ holds, equation (9.12) tell us that we only need to prove that

$$(9.16) \quad t \leq s - u$$

If s and t have the same parity, $u = \frac{s-t}{2}$ and (9.16) is equivalent to

$$t \leq s$$

which we are assuming.

If s and t have different parity, $u = \frac{s-t+1}{2}$ and (9.16) is equivalent to

$$t + 1 \leq s$$

which is a consequence of $t \leq s$ and the fact that the parities are distinct. Thus, (9.15) is true. This completes the proof of (9.6) and (9.7)

At last, for (9.3), (9.4), we write

$$\begin{aligned} S_1(p, q, m_2; b, c, d) &= \sum_{\alpha, \beta=1}^p \sum_{h=1}^{p-1} \left(\frac{h}{p} \right) e \left(\frac{b\alpha + c\beta + d\bar{q}(m_2\alpha^2\beta - h)}{p} \right) \\ &= \left(\sum_{h=1}^{p-1} \left(\frac{h}{p} \right) e \left(\frac{-d\bar{q}h}{p} \right) \right) S_2(p, q, m_2; b, c, d), \end{aligned}$$

and the result follows from (9.5) and the well-known bound for the Gauss sum, except, possibly, when $d = p$. But in this case,

$$\sum_{h=1}^{p-1} \left(\frac{h}{p} \right) e \left(\frac{-dm_2\bar{q}h}{p} \right) = \sum_{h=1}^{p-1} \left(\frac{h}{p} \right) = 0,$$

this gives (9.3) and thus, it also gives (9.4) in this case. \square

REFERENCES

- [1] V. Blomer: *The average value of divisor sums in arithmetic progressions*, Quart. J. Math. **59** (2007), 275-286.
- [2] J. Brüdern, A. Granville, A. Perelli, R. C. Vaughan and T. D. Wooley: *On the exponential sum over k -free numbers*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **356** (1998), 739-761.
- [3] M.J. Croft: *Squarefree numbers in arithmetic progressions*, Proc. London Math. Soc. **30** (1975), 143-159.
- [4] É Fouvry, S. Ganguly, E. Kowalski and Ph. Michel: *Gaussian distribution for the divisor function and Hecke eigenvalues in arithmetic progression*, to appear, Commentarii Mathematici Helvetici.
- [5] R. R. Hall: *Squarefree numbers in short intervals*, Mathematika, **29**, (1982), 7-17.
- [6] D. R. Heath-Brown: *The square sieve and consecutive square-free numbers*, Math. Ann. **266** (1984), 251-259.
- [7] C. Hooley: *A note on square-free numbers in arithmetic progressions*, Bull. Lond. Math. Soc. **7** (1975), 133-138.
- [8] H. L. Montgomery *Primes in arithmetic progressions*, Michigan Math. J. **17** (1970), 33-39.
- [9] F. Pillichshammer *Euler's constant and averages of fractional parts*, Amer. Math. Monthly **117** (2009), 78-83.
- [10] T. Reuss: *Pairs of k -free Numbers, consecutive square-full Numbers* (arXiv:1212.3150v1 [math.NT])
- [11] P. Shiu: *A Brun-Titchmarsh theorem for multiplicative functions*, J. reine angew. Math. **313** (1980), 161-170.
- [12] G. Tenenbaum: *Introduction to analytic and probabilistic number theory*, Translated from the second French edition (1995) by C. B. Thomas, Cambridge Studies in Advanced Math., 46, (Cambridge University Press, Cambridge, 1995)
- [13] E. C. Titchmarsh: *The Theory of the Riemann zeta-function* (Oxford University Press, 1951).
- [14] K.-M. Tsang, *The distribution of r -tuples of square-free numbers*, Mathematika, **32** (1985), 265-275.
- [15] R. C. Vaughan: *A variance for k -free numbers in arithmetic progressions*, Proc. London Math. Soc. **91** (2005), no. 3, 573-597.
- [16] R. Warlimont: *Über die kleinsten quadratfreien zahlen in arithmetischen Progressionen mit primen Differenzen*, J. reine angew. Math. **253** (1972), 19-23.
- [17] R. Warlimont: *Squarefree numbers in arithmetic progressions*, J. London Math. Soc. (2) **22** (1980), 21-24.

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